Tranching and Rating

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Abstract

In this paper we analyze the source and level of the marketing gains when structured debt securities are sold at yields that reflect only their credit ratings, or specifically at the yield on an equivalently rated reference bond. We distinguish between credit ratings that are based on probabilities of default and ratings that are based on the expected default losses. We show that the marketing gain from subdividing a bond issued against given collateral into subordinated tranches can yield significant profits under the hypothesized pricing system. Increasing the systematic risk or reducing the idiosyncratic risk of the bond collateral increases the profits further. Given a fixed issue size the marketing gain is increasing in the number of tranches.

JEL: G12, G13, G14, G21, G24.

Keywords: Credit Ratings, Collateralized Debt Obligations, expected loss rate, default probability, systemic risk.

1 Introduction

Approximately $471 billion of the $550 billion of collateralized debt obligations (CDOs) that were issued in 2006 were classified by the Securities Industry and Financial Markets Association (SIFMA) as 'Arbitrage CDOs'.[1] These are defined by SIFMA as an 'attempt to capture the mismatch between the yields of assets (CDO collateral) and the financing costs of the generally higher rated liabilities (CDO tranches).'[2] In the simple world of Modigliani and Miller (1958) such arbitrage opportunities would not exist, and Grinblatt and Longstaff (2000) have shown that there are essentially no arbitrage opportunities in the related market for stripped treasury securities. This raises the question on the sources of the arbitrage gains in the markets for CDO's and other structured bonds.[3] In this paper we present a simple theory of the effect of tranching debt and of collateral diversification on the prices at which debt securities can be marketed. The theory can account for the apparent arbitrage opportunities that was offered by the market for CDOs and the explosive growth of the structured finance market in the recent past. Of course, the long-term existence of untranched securitisations such as mortgage backed securities suggests that there are sources of the marketing gains other than the one we consider, such as liquity enhancement.[4]

Our theory rests on the assumption that some investors are not able to assess for themselves the value of the debt securities issued by special purpose vehicles, but must rely instead on credit ratings provided by third parties. We shall make the extreme assumption that securities can be sold in the primary market at yields that reflect only their ratings. This is not to say that all investors rely only on credit ratings - but that at least some do, and that if ratings based valuations exceed fundamental values, then the investment banker will be able to sell to

[1] The remaining issuance is classified as 'Balance Sheet' CDOs which 'remove assets or the risk of the assets off the balance sheet of the originator'.

[2] SIFMA, January 2008.
`http://archives1.sifma.org/assets/files/SIFMA_CDOIssuanceData2007q1.pdf`

[3] We generally follow the terminology of SIFMA and use the generic term CDO to refer to credit instruments issued against a portfolio of other credit instruments. There is a wide variety of CDO types which is discussed in more detail below.

[4] See Subrahmanyam (1991) for a formal model.

these investors in the primary market at prices that depend only on ratings. Our assumption is justified by the attention focused on the role of ratings in the marketing of tranched securities.[5] Though this is denied by the rating agencies, it has been suggested that rating agencies assist in the design of new securities to ensure that they achieve targeted rating assessments.[6] According to the *Financial Times* of December 6, 2007, 'for many investors ratings have served as a universally accepted benchmark', and 'some funds have rued their heavy dependence on ratings'. Even regulators rely on the reports of the rating agencies: "As regulators, we just have to trust that rating agencies are going to monitor CDOs and find the subprime," said Kevin Fry, chairman of the Invested Asset Working Group of the U.S. National Association of Insurance Commissioners. "We can't get there. We don't have the resources to get our arms around it." (International Herald Tribune, June 1, 2007.)

We do not argue that the marketing story we tell is the only explanation for the tranching of debt contracts.[7] Previous contributions rely on asymmetric information and the ability of the issuer either to signal the quality of the underlying assets by the mix of securities sold,[8] or on the differential ability of investors to assess complex risky securities. In Boot and Thakor (1993) cash flow streams are marketed by dividing them and allocating the resulting components to information insensitive and sensitive (intensive) securities. The former are marketed to uninformed investors, and the latter to information

[5] The Treasurer of the State of California recently claimed that "If the state of California received the triple-A rating it deserved, we could reduce taxpayers' borrowing costs by hundreds of millions of dollars over the 30-year term of the still-to-be issued bonds.." Reuters, March 12, 2008. Moody's has agreed to provide municipalities with the equivalent of a corporate bond rating from May 2008; prior to this date default losses for municipal bonds were significantly below those of equivalently rated corporate bonds.

[6] The ambiguities in the relation between the issuer and the rating agency are captured in a publication of Standard & Poor's: 'Either an issuer or an investment bank as the arranger presents a proposed structure. The rating analysts give their preliminary views as to what the rating will be, based upon our published criteria. The arranger in response may change aspects of the transaction. On unusual or novel types of transactions, this process may involve additional dialogue... It's important to re-iterate that in no way what occurs in the structured finance ever amount to "advisory" work.' Standard and Poor's (2007)

[7] Ross (1989) has previously drawn attention to the marketing role of the investment banker for an institution that wishes to sell off some of its low grade assets. However, he does not include the role of the credit rating agencies in his consideration.

[8] Brennan and Kraus (1987), De Marzo and Duffie (1999), DeMarzo (2005).

gathering specialist.[9] Other explanations include what Ross (1989) refers to as the 'old canard' of spanning.[10]

Our analysis is concerned with the limitations of a bond rating system which relies only on assessments either of default probabilities or of expected default losses. It is straightforward to show that a system which relies only on default probabilities is easy to game, e.g. by selling securities with lower recovery rates and holding securities of the same rating but with a higher recovery rate. Only slightly more subtly, a system which relies on expected default losses is also easy to game. This is because a simple measure of expected default loss takes no account of the states of the world in which the losses occur. The investment banker could profit by selling securities whose default losses are allocated to states with the highest state prices per unit of probability.[11] Rating agencies, by providing information about default probabilities or expected default losses, are providing information about the *total risk* of the securities. Although it has been well known for many years that equilibrium values must depend on measures of systematic rather than total risk, this insight has not so far affected the practices of the credit rating agencies. The failure of the credit agencies to recognize the distinction between total and systematic risk creates an arbitrage opportunity for investment banks to exploit the system by selecting collateral characteristics so as to raise the systematic risk of the securities they issue above that of equivalently rated corporate securities with similar (total) default risk. We emphasize that our analysis does not rest on any assumption of bias or inaccuracy in the default probability and loss assessments which underly the ratings assigned by the agencies.

We assume that the underlying collateral against which the structured debt claims are written is properly valued. We also assume that bond ratings are calibrated with respect to single debt claims issued by a reference firm with certain risk characteristics. We then show that under a rating system that is based on default probabilities (e.g. Standard & Poor's and Fitch) or expected default

[9] See also Plantin (2004) and Riddiough (1997).
[10] See Gaur *et al.* (2004).
[11] Coval et al (2007) make a similar point.

losses (e.g. Moody's), the optimal strategy for the issuer is to maximize the number of differently rated tranches. If the risk characteristics of the collateral can be chosen, then the issuer will maximize beta and minimize idiosyncratic risk. A rating system that is based on expected losses (e.g. Moody's) reduces, but does not eliminate all of, the pricing anomalies and the issuer's marketing gains.

Our analysis is most closely related to that of Coval *et al.* (2007) who show that it is possible to exploit investors who rely on default probability based ratings for pricing securities, by selling them bonds whose default losses occur in high marginal utility states. However, unlike our study, their theory has no explicit role for debt tranching. They use a structural bond pricing model to predict yield spreads on CDX index tranches and conclude that there is severe market mispricing: the market spreads are much too low for the risk of the tranches, and this is particularly true for the highly rated tranches. In contrast, our model suggests that highly rated tranches will be subject to the least mispricing, and that the highest marketing gains will come primarily from the junior tranches.[12]

Other important contributions in CDO pricing, which are not directly related to our study, include Longstaff and Rajan (2007) who estimate a multinomial Poisson process for defaults under the risk neutral density from the prices of CDO tranches, and Firla-Cuchra (2005) who provides empirical evidence on the determinants of initial offering spreads on structured bonds.

An important implication of the fact that tranched securities are typically written against diversified portfolios of securities is that defaults of tranched securities of a specified rating will tend to be much more highly correlated than defaults of securities of the same rating issued by a typical undiversified firm - in the limit the defaults of the tranched securities will be perfectly correlated. This, together with the systematic event of a decline in underwriting standards and a bubble in house prices, accounts for the fact that we see almost all

[12] An important difference between our analysis and that of Coval *et al.* is that while they assume an exogenous fixed recovery rate in the event of default, we allow the recovery rate to depend on the value of the underlying assets.

highly rated securities issued against portfolios of subprime mortgages made in 2006 and 2007 experiencing ratings deterioration at the same time. This has profound implications for regulatory systems for bank capital that depend on bond ratings.[13] A portfolio of n A rated CLO tranches will in general be much more risky than a portfolio of n A rated bonds issued by corporations. However, an analysis of the regulatory implications of credit rating systems is beyond the scope of this paper. But our analysis has implications also for the emerging debate as to whether structured products should be rated on a different scale from other credit instruments.[14]

Section 2 provides an introduction to the market for structured bonds and Section 3 discusses credit ratings and the market for CDO's. Section 4 presents a general analysis of the investment banker's problem of security design and characterizes his marketing profit. Section 5 introduces our simple analytical model of rating yields within the context of the CAPM and the Merton model of debt pricing. In section 6 the marketing gains from tranching corporate debt issues are analysed. In Section 7 the model is extended to case of a securitisation of corporate bonds.

2 Structured Bonds

In 1970 the U.S. Government National Mortgage Association (GNMA) sold the first securities backed by a portfolio of mortgage loans. In subsequent years GNMA further developed these securitisation structures and through which portfolios of commercial or residential mortgages are sold to outside investors. From the mid 1980s the concept was transfered to other asset classes such as auto loans, corporate loans, corporate bonds, credit card receivables, etc. Since

[13]Under Basel 1 the regulatory capital requirement was independent of the creditworthiness of the borrower. Under Basel II capital requirements depend either on external ratings, as discussed here, or on an approved internal rating system, which takes default probabilities and expected losses in case of default into account. Global regulators are re-examining the degree to which regulatory frameworks have become dependent of credit ratings.*Financial Times* June 12, 2008.

[14]'a managing director at Moody's said: "we did go out and ask the community whether they wanted a different category of rating (for structured products) because this idea was floated by regulators but the strong response was please don't change anything.' *Financial Times* June 11, 2008.

then the market for the so called asset backed securities (ABS) has seen tremendous growth. According to the Bank of England (2007) the global investment volume in the ABS market was USD 10.7 trillion by the end of 2006.

In a securitisation transaction a new legal entity, a Special Purpose Vehicle (SPV), is created to hold a designated portfolio of assets. The SPV is financed by a combination of debt and equity securities. A key feature is the division of the liabilities into tranches of different seniorities: payments are made first to the *senior* tranches, then to the *mezzanine* tranches, and finally to the *junior* tranches. This prioritization scheme causes the tranches to exhibit different default probabilities and different expected losses. While the super-senior tranche is almost safe, the junior tranches bear the highest default risk.[15]

Typically the SPV issues two to five rated debt tranches and one non-rated equity or first loss piece (FLP).[16] In an empirical study of European securitisation transactions, Cuchra and Jenkinson (2005) found that a rather high percentage of the total portfolio volume is sold in tranches with a rating of A or better (on average 77%). AAA tranches on average accounted for 51% of the transaction but with a high variation across transactions types (between 30% and 89%). As shown by Franke *et al.* (2007) the size of the FLP varies significantly across transactions - from 2% to 20% in their sample of European CDOs.

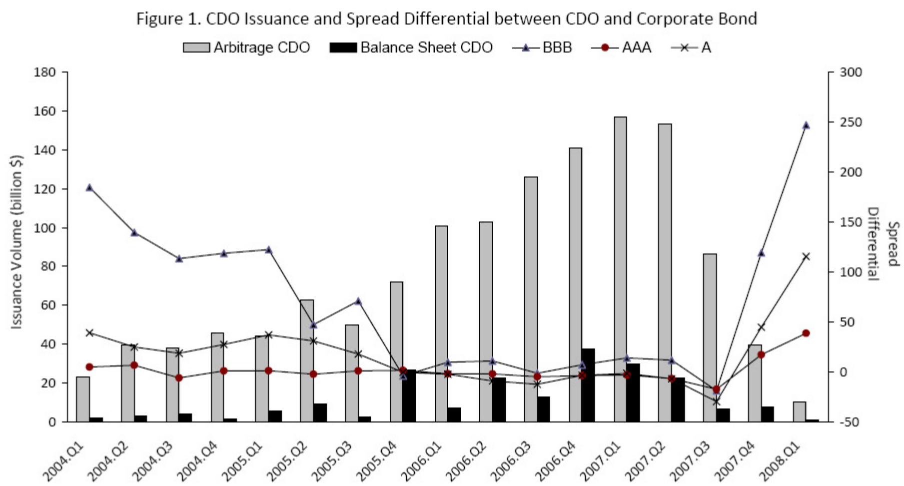
The originator of the CDO specifies in advance the number of tranches and their desired ratings. Due to information asymmetries between the originator and the investors concerning the quality of the underlying portfolio, the tranches need to be rated by an external rating agency. After analyzing the transaction using cashflow simulations and stress testing,[17] two or three of the leading rating agencies assign ratings to the tranches. These ratings reflect the tranches' default probability (Standard & Poor's and Fitch) or expected default losses

[15]This is only a very brief and simplified description of these transactions. For a more detailed discussion on securitisation structures see Hein (2007).

[16]Ashcraft and Schuerman (2008) describe a vehicle whose liabilities were dividend into 16 tranches with 12 different credit ratings.

[17]Beside this quantitative analysis, which plays a major role in the rating process, rating agencies also take into account qualitative aspects such as the servicer's, asset manager's and trustee's skills and reputation as well as legal aspects.

Figure 1:



(Moody's), and are used by investors as an indicator of the tranche's quality.

Figure 1 displays the quarterly issuance volumes of balance sheet and arbitrage CDOs from 2004 to the first quarter of 2008 as reported by SIFMA. Total issuance of CDO's exploded in the years leading up to the sub-prime crisis with total quarterly issuance rising from \$25.0 billion in the first quarter of 2004 to \$178 billion in the fourth quarter of 2006. Even more significant is the fact that most of the growth in CDO offerings came from 'arbitrage' CDO's which SIFMA describes as motivated by mismatches between yields on the collateral and the average yields on the liability tranches sold against the collateral. It is the role of credit ratings in creating this mismatch that is the focus of our analysis.

Figure 1 also shows the spread differential between CDO tranches and equivalently rated corporate bonds.[18] Along side the enormous growth in CDO issuance volumes, we see a sharp decline in the spread differential for different

[18]To derive the spread differential we take average tranche spreads on European CLOs as reported by HSBC Global ABS Research and subtract corporate bond spreads of the same rating class. The tranche spreads are quoted over EURIBOR/LIBOR since the tranches are floating rate notes. The corporate bond spreads are derived by comparing yields on the corresponding iBoxx Corporate (AAA/A/BBB) index to the iBoxx Sovereign index with the same maturity.

rating classes, especially for the BBB grade. From the first quarter of 2005 up to the third quarter of 2007, just before the subprime crisis, the spread differential was negligible with tranche spreads being even slightly below those of equivalently rated corporate bonds. During that period, the spread on AAA (A) rated tranches was, on average, 4.75 (8.19) basis points smaller than the corresponding bond spreads. From the beginning of the subprime crisis in mid 2007 issuance volumes dried out and the spread differentials sharply increased.

3 Credit Ratings

Seven rating agencies have received the Nationally Recognized Statistical Rating Organization (NRSRO) designation in the United States, and are overseen by the SEC: Standard & Poor's , Moody's, Fitch, A. M. Best, Japan Credit Rating Agency, Ltd., Ratings and Investment Information, Inc. and Dominion Bond Rating Service. The three major rating agencies, S&P, Moody's, and Fitch, dominate the market with approximately 90-95 percent of the world market share. Moody's ratings are based on estimates of the expected losses due to default, while S&P and Fitch base their ratings on estimates of the probability that the issuing entity will default.[19]

Standard and Poor's ratings for structured products have broadly the same default probability implications as their ratings for corporate bonds.[20] Before 2005 the implied default probabilities for corporate and structured product ratings were the same. In 2005 corporate ratings were "delinked from CDO rating quantiles" in order to "avoid potential instability in high investment-grade scenario loss rates". As a result, "CDO rating quantiles are higher than the corporate credit curves at investment grade rating levels, and converge to the corporate credit curves at low, speculative-grade rating levels" now.[21]

[19] S&P explicitly state that 'Our rating speaks to the likelihood of default, but not the amount that may be recovered in a post-default scenario.' Standard and Poor's (2008).

[20] For Standard and Poor's at least, the rating assigned to a particular tranche does not depend upon the size of the tranche, but only on the total face value of the tranche and tranches that are senior to it: "Tranche thickness" generally does not affect our ratings, nor their volatility, since our ratings are concerned with whether or not a security defaults, *not how much loss it incurs in the event of default.*' Standard and Poor's (2007).

[21] See Standard & Poor's (2005).

Thus, in 2005 S&P liberalised the ratings for structured bonds. Table 1 shows cumulative default frequencies for corporate bonds by rating and maturity as reported by Standard and Poor's (2005), and Table 2 shows the cumulative default frequency for CDO tranches. For example, the five year cumulative default probability implied by a B rating for a CDO tranche is now 26.09 percent as compared with 24.46 percent for a corporate bond. If the investors are aware of the different implied default rates implied by the same rating for corporate bonds and CDO tranches, then we should expect the tranches to sell at higher yields for this reason alone.

Moody's ratings for both corporate and structured bonds are based on the cumulative 'Idealized Loss Rates' which are shown in Table 3. According to Moody's, 'the idealized loss rate tables were derived based on a rough approximation of the historical experience as observed and understood as of 1989. In addition we assumed extra conservative (low) loss rates at the highest rating levels...we use the idealzed loss rates to model the ratings.'[22] Although it would seem more reasonable to base credit ratings on expected default losses rather than simply on default probability, Cuchra (2005, p 16) reports that in European markets for structured finance 'S&P ratings explain the largest share of the total variation in (new issue) spreads, followed by Moody's and Fitch.'

4 Theoretical Framework of Rating Based Pricing and Tranching

Among the primary roles of the investment banker are the marketing of new issues of securities, and the provision of advice on the appropriate mix of securities to finance a given bundle of assets. Although the classic Modigliani and Miller (1958) analysis of capital structure implies that all financing mixes are equally good, it is now recognized that the mix of securities sold may be important for valuation on account of control, incentive, tax, liquidity, information, and bankruptcy cost considerations, and advice on these issues provides a legitimate role for the investment banker. However, apart from liquidity and

[22] Private communication from Moody's.

information, none of these factors offers any *direct* connection between the mix of securities and the valuation of given cash flow streams. In our model the marketing gains from the choice of the financing mix arise from the difficulty in evaluating the different cash flow claims in the capital structure of a structured bond issuer or of an SPV which holds the collateral in case of a securitisation. This forces many investors to rely on credit ratings as the sole basis of their evaluation and, as mentioned above, these ratings do not reflect the *systematic risk* characteristics of the securities being rated.

4.1 A Simple Model of Ratings Based Pricing

The importance of credit ratings for the pricing of structured bonds is documented by Cuchra (2005) who shows that 'the relation between price and credit rating for each tranche is very close indeed and consistent across all types of securitisations ... this relationship seems considerably stronger than in the case of corporate bonds.'[23] This motivates our fundamental assumption, that investment bankers are able to sell new issues of structured bonds at yields to maturity that are the same as the yields on equivalently rated bond issued by a reference firm.[24] The main difference between these two types of security is that the reference bond is secured by the assets of a single firm and represents a senior claim with respect to equity, whereas the structured bond is either a subordinated bond within a tranched debt structure of a single firm or a tranche that is secured by a portfolio of bonds which is divided into tranches of different seniorities.

Throughout this section, we shall use an asterisk to denote variables that correspond to the rating agency's reference bond or its issuer, and use the same variables without the asterisk to denote the corresponding variable for the structured bond or its issuer. Thus, let W_k^* and W_k denote the values of

[23] Cuchra (2005, p2) also remarks that 'the tranche-specific, composite credit rating ... is the primary determinant of (launch) spreads.

[24] This assumption also seems to be consistent with the expectations of the rating agencies. For example, 'Do ratings have the same meaning across sectors and asset classes? The simple answer is "yes". Across corporates, sovereigns and structured finance, we seek to ensure to the greatest extent possible that the default risk commensurate with any rating category is broadly similar.' Standard and Poor's (2007). Similarly, the 'idealized loss rates' to which Moody's structured product ratings are calibrated are taken from corporate bond experience.

pure discount debt securities with face values B_k^* and B_k, rating k, and maturity τ[25] when issued by the reference firm with asset value, V^*, and an arbitrary corporate bond issuer or an SPV holding collateral with asset value V.

Let y_k^* denote the yield to maturity, and $\phi_k^* \equiv W_k^*/B_k^* = e^{-y_k^* \tau}$ the ratio of the market value of a k rated pure discount corporate bond to its face value when issued by the reference firm. Let S_k denote the *sales price* of a pure discount structured debt security with nominal value B_k and rating k issued by an arbitrary structured bond issuer or an SPV. Our assumption is that the *sales price*, S_k, at which a new debt security can be sold, bears the same relation to its face value as does the value of an equivalently rated debt security with the same maturity issued by the reference firm:

Pricing Assumption:

$$S_k = \phi_k^* B_k = e^{-y_k^* \tau} B_k.$$

Let P^* denote the *physical* probability distribution of the asset value of the reference firm at the maturity of the bond, and let P denote the corresponding probability distribution for the corporate bond issuer or of the collateral held by the SPV. The price of any contingent claim written on the value of the corporation, V^*, or the value of the structured bond collateral, V, can be expressed as the discounted value of the contingent claim payoff under the equivalent martingale measures Q^* and Q. The link between the physical and risk neutral measures is given by the conditional pricing kernels for contingent claims on the underlying assets, $m^*(v)$ and $m(v)$, with $f_{Q^*}(v) = m^*(v) f_{P^*}(v)$ and $f_Q(v) = m(v) f_P(v)$, and $f(v)$ is the density function of the terminal underlying asset value v under the corresponding measure.

We consider two different rating systems:

(i) *Default Probability Based Rating*

The bond rating, k, is a monotone decreasing function of the probability of default, $\mathcal{R}_{\mathcal{P}}(\Pi)$, $\mathcal{R}'_{\mathcal{P}}(\Pi) < 0$.

[25] For simplicity we will drop the maturity subscript τ in the following.

(ii) *Expected Default Loss Based Rating*

The bond rating, k, is a monotone decreasing function of the expected default loss, $\mathcal{R}_{\mathcal{L}}(\Lambda)$, $\mathcal{R}'_{\mathcal{L}}(\Lambda) < 0$.

We assume for simplicity that all defaults take place at maturity, and denote the default loss rate for a bond with rating k and maturity τ, by Λ_k, and denote the probability of default by Π_k. The probabilities of default and the expected default loss rates are determined by the *physical* probability distributions, P and P^*, while the market values of the instruments, and therefore the ratios of market value to the nominal payments, are determined by the promised nominal payments and the risk neutral probability distributions, Q and Q^*, as illustrated below:

Agency Rated Reference Bond:

$$\Lambda_k, \Pi_k \quad \overset{P_k^*}{\longleftarrow} \quad B_k^* \quad \overset{Q_k^*}{\longrightarrow} \quad \frac{W_k^*}{B_k^*} \equiv \phi_k^* = e^{-y_k^* \tau}$$

Agency Rated Structured Bond:

$$\Lambda_k, \Pi_k \quad \overset{P_k}{\longleftarrow} \quad B_k \quad \overset{Q_k}{\longrightarrow} \quad \frac{W_k}{B_k} \equiv \phi_k = e^{-y_k \tau}$$

Thus the fair *market value* of the structured bond is:

$$W_k = \phi_k B_k = e^{-y_k \tau} B_k$$

which usually differs from the ratings based *sales price* as defined before. In effect, we assume that the investment banker is able to sell the security at a price that reflects the risk neutral probability distribution, Q_k^*, that is appropriate for a typical corporate issuer of a bond with the same probability of default or expected loss.

First we consider the gains from rating based pricing and tranching within a general model of valuation. In our subsequent analysis, we present a parametric model of the marketing gains that the issuer can reap from (i) differences between the physical probability distributions of the reference firm and that of the structured bond issuer; (ii) differences between the risk neutral probability distributions; (iii) issuing tranched debt when there are different physical or risk neutral distributions.

4.2 Issuing a Single Bond

As a starting point, we characterize the marketing gain from rating based pricing when issuing a single bond against the assets of a single firm or against a portfolio of assets. When ratings are based on *default probability*, the face value of the bond with rating k issued by the reference firm and the face value of the single debt tranche with the same rating k issued by an arbitrary firm or an SPV are defined by

$$\int_0^{B_k^*} f_{P^*}(v)dv = F_{P^*}(B_k^*) = \Pi_k = F_P(B_k) = \int_0^{B_k} f_P(v)dv \tag{1}$$

where F_{P^*} (F_P) denotes the *cdf* with respect to the physical probability measure P^* (P) and f_{P^*} (f_P) are the corresponding density functions.

When ratings are based on *expected default loss*, the face values are defined by

$$\Lambda_k = \frac{\mathcal{L}^*}{B_k^*} = \frac{\mathcal{L}}{B_k} \tag{2}$$

with

$$\mathcal{L}^* = \int_0^{B_k^*} (B_k^* - v) f_{P^*}(v)dv \tag{3}$$

$$\mathcal{L} = \int_0^{B_k} (B_k - v) f_P(v)dv \quad . \tag{4}$$

The marketing gain, $\mathbf{\Omega}$, from issuing the security is equal to the difference between the sales price, S_k, and the market value W_k:

$$\begin{aligned} \mathbf{\Omega} &= S_k - W_k \\ &= [\phi_k^* - \phi_k]\, B_k \end{aligned} \tag{5}$$

Setting the interest rate equal to zero for simplicity, the value of the new security is given by:

$$\begin{aligned} W_k &= \int_0^{B_k} v f_Q(v)dv + B_k \int_{B_k}^{\infty} f_Q(v)dv \\ &\equiv \phi_k B_k \end{aligned} \tag{6}$$

Similarly, $\phi_{k,\tau}^*$ is defined implicitly by the valuation of the corporate liability:

$$\begin{aligned} W_k^* &= \int_0^{B_k^*} v f_{Q^*}(v)dv + B_k^* \int_{B_k^*}^{\infty} f_{Q^*}(v)dv \\ &\equiv \phi_k^* B_k^* \end{aligned} \tag{7}$$

Combining (6) and (7) with (5), the marketing gain may be written as:

$$\begin{aligned}\mathbf{\Omega} &= B_k \left\{ \frac{1}{B_k^*} \int_0^{B_k^*} v f_{Q^*}(v) dv + \int_{B_k^*}^{\infty} f_{Q^*}(v) dv \right\} \\ &- B_k \left\{ \frac{1}{B_k} \int_0^{B_k} v f_Q(v) dv + \int_{B_k}^{\infty} f_Q(v) dv \right\} \end{aligned} \tag{8}$$

where B_k^* and B_k are given by equation (1) under a default probability rating system, and by equation (2) under a default probability rating system. Sufficient conditions for the marketing gain to be positive or negative are given in the following Lemma:

Lemma 1 *Default Probability Rating System*

(a) *The marketing gain, $\mathbf{\Omega}$, will be positive if P first order stochastically dominates P^* ($P \geq^{FSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the marketing gain will be negative if $P^* \geq^{FSD} P$ and $Q \geq^{SSD} Q^*$.*

(b) *Moreover if two corporate issuers or two SPVs have the same risk-neutral distribution Q and their physical distributions, P_1 and P_2, are such that $P_2 \geq^{FSD} P_1 \geq^{FSD} P^*$, and $Q^* \geq^{SSD} Q$, then the marketing gain from issuing a structured bond with a given rating k will be greater for the second issuer (SPV_2) than for the first issuer (SPV_1).*

Proof: See Appendix

Lemma 2 *Expected Default Loss Rating System*

(a) *The marketing gain, $\mathbf{\Omega}$, will be positive if P second order stochastically dominates P^* ($P \geq^{SSD} P^*$) and Q^* weakly dominates Q by Second Order Stochastic Dominance ($Q^* \geq^{SSD} Q$). Conversely, the marketing gain will be negative if $P^* \geq^{SSD} P$ and $Q \geq^{SSD} Q^*$.*

(b) *Moreover if two corporate issuers or two SPVs have the same risk-neutral distribution Q and their physical distributions, P_1 and P_2, are such that $P_2 \geq^{SSD} P_1 \geq^{SSD} P^*$ and $Q^* \geq^{SSD} Q$, then the marketing gain from issuing a structured bond with a given rating k will be greater for the second issuer (SPV_2) than for the first issuer (SPV_1).*

Proof: See Appendix

As a direct application of part (a) of Lemma 1, consider the situation in which either the single period CAPM or its continuous time version holds, and V and V^* have the same total risk. The risk neutral measures will then be identical:

$Q \equiv Q^*$. P will first order stochastically dominate P^* whenever the structured bond issuer has a beta coefficient higher than that of the reference firm because this will imply a higher mean return for the structured bond issuer. Part (b) of Lemma 1 implies that, for a given total risk and bond rating, the marketing gain will be monotonically increasing in the beta of the structured bond collateral.

4.3 Issuing Multiple Tranches

Lemmas 1 and 2 characterize conditions under which the marketing gain from a single debt issue is positive given our pricing assumption. However, some corporations also issue several subordinated debt tranches and also most asset securitisations involve multiple tranches.[26] In this section we consider when the marketing gain can be increased by issuing additional tranches. To analyze the gains from introducing multiple tranched securities, consider the gain from replacing a single debt issue with face value B_k and rating k with two tranches. Denote the face value of the senior tranche by B_{1,k_1} and its rating by k_1, and denote the face value of the junior tranche by $B_{2,k_2} \equiv B_k - B_{1,k_1}$ and its rating by k_2.[27]

Under a *default probability* rating system, the default probability of the single tranche, Π_k, is equal to the default probability of the junior tranche of the dual tranche structure, since in both cases the SPV defaults when its terminal value, V, is less than $B_k = B_{1,k_1} + B_{2,k_2}$. Hence, under the rating based pricing the junior tranche sells at the same (corporate bond) yield as the single tranche: $\phi^*_{k_2} = \phi^*_k$. On the other hand, the senior tranche has a lower default probability than the single tranche issue so that it sells at a lower yield such that $\phi^*_{k_1} > \phi^*_k$, and the extra gain from switching from a single-tranche to a two-tranches structure is $(\phi^*_{k_1} - \phi^*_k)B_{1,k_1}$. It is straightforward to extend this argument to additional tranches as stated in the following lemma:

Lemma 3 *Default Probability Rating System*

[26] Cuchra and Jenkinson (2005) report that in 2003 the average number of tranches in European securitisations was 3.93 and in US securitisations 5.58.

[27] Note that in our notation, B_{j,k_j}, j denotes the seniority of the tranche issued and k_j denotes its rating. Note that neither the payoff nor the rating of a given tranche depend on the existence or characteristics of more junior tranches.

Under a default probability rating system it is optimal to subdivide a given tranche into a junior and a senior tranche with different ratings, whenever the pricing kernel for the reference issuer, $m^*(v)$, is a decreasing function of the underlying asset value.

The Lemma implies that it is optimal to have as many tranches as there are different rating classes.

Lemma 4 *Expected Default Loss Rating System*
Under an expected default loss rating system, if a given tranche is profitable, then it is optimal to subdivide the tranche into a junior and a senior tranche with different ratings, whenever the pricing kernel for the reference issuer, $m^*(v)$, is a decreasing function of the underlying asset value.

Proof: See Appendix

Lemmas 3 and 4 are consistent with the findings of Cuchra and Jenkinson (2005) that the number of tranches in European securitisations has displayed a secular tendency to increase which they attribute to the growing sophistication of investors in these markets, and that securitisations characterized by greater information asymmetry tend to have more tranches with different ratings.

5 Parametric Model of Ratings Yields

In order to quantify the gains from tranching and securitisation when bond issues are made at yields that reflect only their ratings it is necessary to have a model of yields as a function of ratings. We assume that bond ratings are based on the risk characteristics of a reference firm, the value of whose assets (V^*) follows a geometric Brownian motions:

$$dV^* = \mu^* V^* dt + \sigma^* V^* dz^* \tag{9}$$

where $\mu^* = r_f + \beta^*(r_m - r_f)$, r_f denotes the risk-free rate, $(r_m - r_f)$ the excess market return, and β^* the CAPM beta coefficient. The total risk σ^* can be decomposed into a systematic and an unsystematic risk component: $\sigma^* = \sqrt{(\beta^*\sigma_m)^2 + \sigma_\varepsilon^{*2}}$, where σ_m denotes the market volatility and σ_ε^* denotes the residual risk.

When ratings are based on *default probabilities*, the face value of the reference bond with rating k, B_k^*, depends on its default probability Π_k, i.e. the probability that the assets of the reference firm are less than B_k^* at maturity:[28]

$$\Pi_k = \mathcal{N}\left(-\frac{ln(V^*/B_k^*) + (\mu^* - 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}}\right) \tag{10}$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution. Then the face value, B_k, may be expressed as a function of Π_k:

$$B_k^* \equiv \frac{V^*}{exp\{-\mathcal{N}^{-1}[\Pi_k]\sigma^*\sqrt{\tau} - (\mu^* - 0.5\sigma^{*2})\tau\}} \tag{11}$$

When ratings are based on *expected default losses* the face value of a reference bond with rating k, B_k^*, depends on its loss rate Λ_k:

$$B_k^* = \frac{\mathcal{L}_k^*}{\Lambda_k} \tag{12}$$

where the expected default loss, $\mathcal{L}_k^*$, is given by

$$\mathcal{L}_k^* = B_k^*\mathcal{N}(-d_2^{P^*}) - V^*e^{\mu^*\tau}\mathcal{N}(-d_1^{P^*}) \tag{13}$$

with

$$d_1^{P^*} = \frac{ln(V^*/B_k^*) + (\mu^* + 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}} \tag{14}$$

$$d_2^{P^*} = d_1^{P^*} - \sigma^*\sqrt{\tau} = \frac{ln(V^*/B_k^*) + (\mu^* - 0.5\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}}. \tag{15}$$

The market value of the rating k reference bond, W_k^*, is given by the Merton (1974) formula:

$$W_k^* = B_k^*e^{-r_f\tau}\mathcal{N}(d_2^{Q^*}) + V^*\mathcal{N}(-d_1^{Q^*}) \tag{16}$$

where $d_1^{Q^*}$ and $d_2^{Q^*}$ are defined as in equations (14) and (15) substituting r_f for μ^*.

Given the market value and the face value of the reference bond, we get the bond yield for rating class k as

$$\frac{W_k^*}{B_k^*} = \phi_k^* = e^{-y_k^*\tau} \tag{17}$$

[28]For convenience we again drop the maturity subscript τ, although both Π_k and B_k^* depend on the time to maturity.

It is clear that different pairs of μ^* (β^*) and σ^* will lead to different values for W_k^* and B_k^*, and hence ϕ_k^* and y_k^*. This means that the rating based yield is not unique for a given rating class. This is precisely the reason why mispricing errors occur. The mechanisms of mispricing are further elaborated in the next two sections.

6 Marketing Gains from Rating Based Pricing of Corporate Debt

In the following we assume that the asset value of an arbitrary corporate issuer (V) also follows a geometric Brownian motion with parameters (μ, σ), where $\mu = r_f + \beta(r_m - r_f)$.[29]

6.1 Issuing Single Debt

Consider first the case where a single debt security with predetermined credit rating, k, is issued. When ratings are based on *default probabilities* [*expected default losses*], the face value of the bond, B_k, is derived by substituting (V, μ, σ) for the corresponding variables in equation (11) [(13)] as given in the previous section.

Under the rating-based *Pricing Assumption*, the bond is sold at the yield determined by its rating. Hence, the sales price is based on the bond yield as derived in (17):

$$S_k = \phi_k^* B_k \tag{18}$$

Then the marketing gain, $\mathbf{\Omega}$, equals

$$\mathbf{\Omega} = S_k - W_k \tag{19}$$

The marketing gain will depend on the relation between (μ, σ) and (μ^*, σ^*) as discussed in Lemmas 1 and 2. If the parameters of the reference firm and the corporate issuer are the same, i.e. $\mu = \mu^*$ and $\sigma = \sigma^*$, then the marketing gain will be zero.

[29] In contrast to the previous section, the parameter values here do not have an asterisk * which is only used for the reference bond.

6.2 Issuing Multiple Debt Tranches

In considering subordinated issues it is convenient to define B_{k_i}, the cumulative face value, as the sum of the face values of all bonds senior to the bond with rating k_i, including the k_i rated bond itself, so that B_{i,k_i}, the face value of bond i with rating k_i $B_{i,k_i} = B_{k_i} - B_{k_{i-1}}$ where k_{i-1} denotes the rating of the immediate senior bond. The face value of the most senior bond, B_{1,k_1}, is equal to B_{k_1}.

Under a *default probability* rating system, B_{k_i} is derived as before by substituting the appropriate parameters in equation (11).

The calculation of the cumulative face value of subordinated debt is less direct under the *expected default loss* rating system. In this case the expected loss, $\mathcal{L}_{i,k_i}$, on the *ith* bond tranche with face value B_{i,k_i}, is $\mathcal{L}_{i,k_i} = \mathcal{L}_{k_i} - \mathcal{L}_{k_{i-1}}$ with $\mathcal{L}_{k_i}$ and $\mathcal{L}_{k_{i-1}}$ as defined in (13). Hence the expected loss rate on the *ith* bond tranche is:

$$\Lambda_{k_i} = \frac{\mathcal{L}_{i,k_i}}{B_{i,k_i}} = \frac{\mathcal{L}_{k_i} - \mathcal{L}_{k_{i-1}}}{B_{k_i} - B_{k_{i-1}}} \; for \; i > 1 \tag{20}$$

and for the most senior bond

$$\Lambda_{k_1} = \frac{\mathcal{L}_{k_1}}{B_{1,k_1}} = \frac{\mathcal{L}_{k_1}}{B_{k_1}} \tag{21}$$

which corresponds to equation (12). From $\Lambda_{k_1}, \ldots, \Lambda_{k_I}$ the implicit equations for B_{i,k_i}, (20) and (21), may be solved recursively starting with the most senior bond.

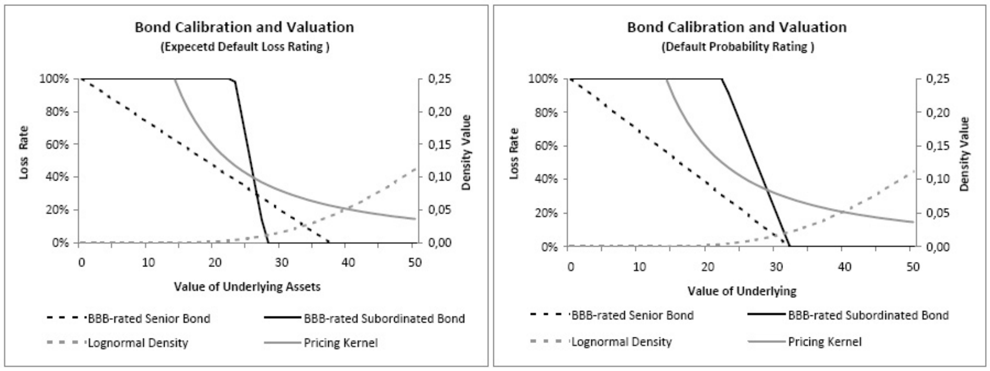
The market value of the *ith* bond tranche with face value B_{i,k_i} is equal to the difference between market values of adjacent cumulative bonds: $W_{i,k_i} = W_{k_i} - W_{k_{i-1}}$ with W_{k_i} and $W_{k_{i-1}}$ as determined in the single bond case.

Using the rating-based *Pricing Assumption*, the sales price of the *ith* bond tranche, S_{i,k_i}, is given by

$$S_{i,k_i} = \phi^*_{k_i} B_{i,k_i} = e^{-y^*_{k_i}\tau} B_{i,k_i} = \frac{W^*_{k_i}}{B^*_{k_i}} B_{i,k_i} \quad . \tag{22}$$

where $y^*_{k_i}$ is derived from the reference bond as described in section 5. Note that $y^*_{k_i} \neq y^*_{i,k_i}$ that is the reference bond yield is calculated based on a single debt

Figure 2: This figure shows the loss rate profile of a BBB rated senior (reference) bond and a BBB rated subordinated bond under both rating system using the same parameter values as in Tables 4 and 5. Additionally, the log-normal density and the pricing kernel for a risk averse investor are given.



issue and applied to equivalently rated subordinated bond within a tranched structure. The marketing gain on the *ith* bond tranche is

$$\mathbf{\Omega}_i = S_{i,k_i} - W_{i,k_i} \quad . \tag{23}$$

The total marketing gain derived from tranching is $\mathbf{\Omega} = \sum_i \mathbf{\Omega}_i$.

6.3 Numerical Examples

In this section we present estimates of the gains to rating based pricing and tranching as described in sections 4.2 and 4.3, assuming a risk-free interest rate of 3.5%, a market risk premium of 7%, and a market volatility of 14%.[30]

Tables 4 and 5 illustrate the pricing of 5-year subordinated bonds under the *default probability* and the *expected default loss* rating systems, respectively, when the asset betas of both the arbitrary corporate issuer and the reference firm is 0.8 and the residual risk, σ_ε, is 25% p.a. Despite the fact that the risk characteristics of the issuer and the reference firm are identical, the gain to

[30]From 1927 to 2007 the US equity market risk premium has averaged about 8.2 percent and the risk-free rate has averaged about 3.8 percent. (see Kenneth R. French Data Library: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. Welch (2000) reports that the arithmetic long-term equity premium consensus forecast is about 7 percent. The marketing gains are increasing in the assumed value of the market risk premium so we are adopting a conservative position. The annualized monthly standard deviation of the Fama-French market factor from January 1946 to March 2008 is 14.5%.)

tranching the debt is 5.45% under the default probability rating system and 0.47% under the expected default loss rating system. Except for the AAA-rated bond, which corresponds to the senior bond of the reference firm, the marketing gain is positive for all subordinated tranches and the profit is largest for the most subordinated bond, i.e. the bond with the lowest rating. As illustrated in Figure 2 these gains are due to the fact that the rating system only accounts for default probabilities. For example, the expected loss of the BBB-rated subordinated bond is much higher than for the equivalently rated senior bond issued by the reference firm. Additionally, the subordinated bond realizes higher loss rates in lower states of the world with a high pricing kernel of a risk-averse investor. The expected default loss rating system is more accurate because it takes account of the magnitude of losses as well as the probability of defaults.

Panel A of Table 4 shows the pricing of the reference bonds and the determination of the ratio of price to face value, ϕ_k^*, under a *default probability* rating system when the Standard & Poor's ratings are used to infer default frequencies. For each of the five reference bonds, the default probability, Π_{k_i}, is taken from Table 1; the face values of the bonds, B_k^*, and the market values, W_k^*, are calculated from equations (11) and (16). The expected default loss rate, which is included for comparison (and is not used in further calculations in this table), is calculated from equation (12).

In Panel B, the fourth column reports the cumulative face value, B_{k_i}, of an untranched bond with probability of default Π_{k_i} issued by the corporation; the cumulative face value is again calculated from equation (11). In this example the face values of the untranched (cumulative) bonds equal those of the correspondingly rated reference bonds because the risk characteristics of the corporate issuer and the reference firm are the same. The face values of the bond tranches are obtained by taking differences of the B_{k_i}.

The market values of the untranched (cumulative) bonds, W_{k_i}, are determined by the Merton formula as given in equation (16), and the market value of the bond tranches, W_{i,k_i}, are obtained as first differences of W_{k_i}. The sales price of bond tranche i, S_{i,k_i}, is obtained by multiplying the face value, B_{i,k_i}, by $\phi^*_{k_i}$, the price to face value ratio for the equally rated reference bond. This ensures that the issue yield to maturity of each tranche is equal to that of the equivalently rated reference bond. Finally, the marketing gain is the difference between the sales price and the market value of each bond tranche. The corporate equity is priced at the equilibrium market value.

The equilibrium yields on the junior bonds in Panel B significantly exceed those of the equivalently rated reference bonds in Panel A, because, although they have the same probability of default, they have much lower expected recovery rates in default. The equilibrium yield on the B rated tranche is 10.93% as compared with 6.31% for the B rated reference bond, which implies a marketing gain of 18.29 - 14.51 = 3.78. The gains on the higher rated tranches are proportionally smaller, and the total gain from securitisation is 5.45. In this example, the gains arise primarily from the junior tranches.

Table 5 displays the calculations under an *expected default loss* rating system using Moody's idealized loss rates. Panel A shows for each rating class the face values and equilibrium yield of the reference bonds. The cumulative face values, B_{k_i}, are derived by iterating equations (20) and (21) using the expected loss rates reported in Table 3. The equilibrium values, W_{k_i}, and price ratios, $\phi^*_{k_i}$, are derived as in Table 4. The probability of default which is included for comparison (and is not used in further calculations in this table) is calculated from equation (10).

Comparing Panel A of Tables 4 and 5 it appears that for a given rating class k_i both the estimated probability of default Π_{k_i} and the estimated expected default loss rate is higher under the Moody's rating system as compared to the S&P rating system. For example, for the B rated bond the S%P cumultaive default probability is 24.46% while our rating yield model implies a default probability of 36.73% for a Moody's B rating. And while Moody's reports an idealized loss rate of 11.39% for the B rated bond, our model yields an expected loss rate of only 6.75% for the S&P B rated bond. As a results the price ratios $\phi^*_{k_i}$ are smaller under the expected default loss rating system.

The face values of the untranched (cumulative) bonds in Panel B of Table 5 are calculated to ensure that the default loss rate for each tranche is equal to Λ_{k_i}.[31] The remaining columns of Panel B are calculated in the same way as for Table 4. As under the default probability rating system, the marketing gain is concentrated in the junior tranches. It is not surprising, that the marketing gain of 0.47% of the value of the collateral under the expected default loss rating system is smaller than the gain of 5.45% under the default probability system which takes no account of the size of losses when they occur.

Table 6 reports marketing gains from pricing debt using rating based yields and from tranching corporate debt into five or six tranches under the two rating systems. The most junior of the five tranches has a S&P (Moody's) BB (Ba) rating and the most junior of the six tranches has a B (B) rating. Unlike Tables 4 and 5, the reference firm and the corporate issuer may not share the same risk characteristics, i.e. it is possible that $\beta^* \neq \beta$ and/or $\sigma_\varepsilon^* \neq \sigma_\varepsilon$. Panel A of Table 6 shows, for different values of $(\beta, \sigma_\varepsilon)$ and $(\beta^*, \sigma_\varepsilon^*)$, the total amount of debt issue that the rating can support, the marketing gains for both the five and six tranched debt issue $(\Omega_{BB}^M, \Omega_B^M)$, and the gains from issuing just a single tranche with the same total market value $(\Omega_{BB}^S, \Omega_B^S)$. The difference between $\Omega_\bullet^M$ and $\Omega_\bullet^S$ is the additional gain through tranching the single debt issue into multiple tranches.

In Table 6 the total amount of debt supported by the rating, increases as the systematic risk, β, increases and as the residual risk of the corporation, σ_ε, decreases. In Panel A, cases predicted by Lemma (1a) as unprofitable are marked by 'x', and cases predicted as profitable are marked by '✓'. Similarly, for Panel B, cases predicted by Lemma (2a) as unprofitable and profitable are marked by 'x' and '✓', respectively. In all these cases predicted by our lemmas, our predictions are confirmed. For both rating systems $\Omega_\bullet^S > 0$ whenever $\beta \geq \beta^*$ and $\sigma_\varepsilon \leq \sigma_\varepsilon^*$. Sometimes, the mispricing gain is still positive when the two conditions are not satisfied provided that the violation is marginal. Comparing Ω_{BB}^S and Ω_B^S, the marketing gain from issuing a larger amount of lower rated

[31] See equations (20) and (21).

debt in a single tranche is positive when $\beta > \beta^*$, and lower when $\beta^* > \beta$. Varying the risk characteristic of the reference firm instead of those of the corporate issuer, create a reverse impact on the mispricing gains from issuing a single debt tranche.

Consistent with Lemma 3, the marketing gain from replacing the single debt security with multiple debt tranches is always positive, i.e. $\Omega^M_\bullet > \Omega^S_\bullet$, and the gain from issuing six tranches always exceeds that from issuing five tranches ($\Omega^M_B > \Omega^S_{BB}$ and $\Omega^M_B > \Omega^S_{Ba}$). The gain from multiple tranching is increasing in the systematic risk of the issuer, β, and decreasing in the residual risk, σ_ε.

Overall the default probability rating system is adequate when applied to a single corporate debt issue and when the discrepancies between the risk characteristics of the issuer and the reference firm underlying the ratings are small; in this case the gain and losses are generally less than 1 percent. However, when the risk characteristics of the issuer largely deviate from the reference firm gains and losses of 2 to 3 percent are possible. The limitations of the system when applied to subordinated debt issues is apparent in the fact that the gain can be as much as 11.2 percent when the debt is tranched into six separate pieces. The gains from multiple tranching are less sensitive to the risk characteristics of the reference firm.

Comparing the mispricing gains in Panels A and B, the superiority of a system which takes account of the magnitude of losses is apparent. The marketing gains from issuing a single debt claim are less than 1 percent in most cases and also the gains from multiple tranching are moderate and substantially lower. Just as under a default probability rating system, the marketing gains are increasing in systematic risk of the collateral and decreasing in the residual risk, σ_ε.

Comparing the debt levels it is interesting that when issuing five tranches with a Ba-rated junior tranche, the total debt is almost identical to that under the default probability rating system when the junior tranche has a S&P BB rating. However, when issuing a further B tranche Moody's ratings imply debt levels, which are 8 to 11 percentage points lower.

7 Marketing Gains from Corporate Bond Securitisation

In the previous section, we considered a corporate issuer with asset value V who issues tranched debt. In this section we analyze a corporate bond securitisation through an SPV. We proceed by simulating a portfolio of J bonds issued by J identical firms all with underlying asset value process:

$$dV = \mu V dt + \sigma V dz \quad with \quad V(0) = 100 \tag{24}$$

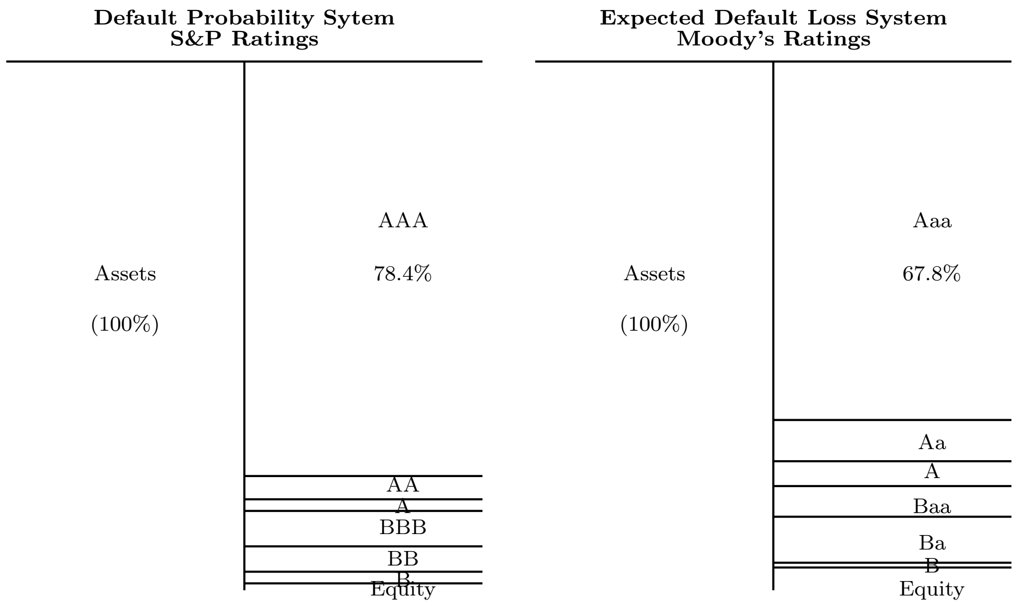
where $\mu = r_f + \beta(r_m - r_f)$ and the total risk σ of each firm can be decomposed into systematic risk, $\beta^2\sigma_m^2$, and idiosyncratic risk, σ_ε. The correlation between the returns on any two firms is $\rho = \frac{\beta^2\sigma_m^2}{\beta^2\sigma_m^2+\sigma_\varepsilon^2}$. Details of the simulation procedure are described in Appendix B. Besides of using the Merton Model with an endogenous recovery rate, we alternatively simulate the securitization by assuming a fixed recovery rate of 40% when bonds in the underlying portfolio default.

Table 7 reports the results for a six tranche securitisations with 125 underlying bonds under the *default probability* and the *expected default loss* rating systems. The parameter values are the same as those used in Tables 4 and 5.[32]

Comparing the tranche structure of the bond securitisation to the debt structure of the single corporate issuer, who issues tranched debt, we see that diversification leads to much higher senior tranches. Figure 3 is a scale model representation of the equilibrium market value capital structures of an SPV for the examples presented in Table 7. Despite the conceptual differences between the Moody's and S&P rating systems, the structures implied by the two systems are fairly similar. The senior tranche is 78.4% of the asset value under the S&P system and 67.8% under the Moody's system, and the equity tranche covers 1.3% of the portfolio volume under the S&P system and 4.34% under the Moody's sytem. The simulated tranche structures correspond to structures observed in the market.

[32]We do not report the valuation of the reference bonds because this is nearly identical to the results shown in Tables 4 and 5.

Figure 3: Equilibrium market value capital structures of an SPV under two different rating systems from Table 7.



The figures in parentheses are the equilibrium market values of each tranche.

In contrast to the case of a single firm issuing tranched debt, we now observe a small, but positive marketing gain on the AAA-tranches although the risk chracteristics of the reference firm and the bonds underlying the SPV are the same. This is again due to risk diversification. The gains on the higher rated tranches are proportionally smaller than those of lower rated tranches, yielding a total gain from securitisation of 4.63 under the default probability rating system and 3.14 under the expected default loss rating system. This contrasts with the suggestion of Coval *et al.* (2007) who claim that 'highly rated tranches should trade at significantly higher yield spreads than single name bonds with identical credit ratings.' Interestingly, this is contradicting by their finding that 'triple-A rated tranches trade at comparable yields to triple-A rated bonds.' which is consistent with our results in Table 7: As derived before the equilibrium yield on the AAA reference bond is 3.51%, while Panel A (B) shows that the equilibrium yield on the AAA tranche is 3.52% (3.51%) under the default probability (expected default loss) rating system. Thus the yield difference on this tranche is only 1 (0) basis point. In contrast the spread between the equilibrium yields on the B tranche and the B corporate bond is 12.57% (7.4%).

Interestingly, under the default probability rating system the total gain from securitising a bond portfolio is smaller than the gain for a single firm (out of this portfolio) issuing tranched debt. This is due to the fact, that the single corporate issuer issues a higher share of lower rated securities whereas due to diversification the SPV issues mostly senior securities, on which not much money can be earned. Concerning the expected default loss rating system, we have a different result. In this case the securitisation of a bond portfolio yields a much higher gain, which shows the enormous effect of risk diversification in addition to tranching. Still the gain under the expected default loss rating system is significantly smaller than under the default probability rating system indicating that it is more accurate.

Departing from our base case scenario, where the SPV portfolio consist of 125 bonds written on firm that has the same risk characteristics, β and $sigma$, as the reference firm, we made a comparative statics analysis by varying different parameters. The simulation results as shown in Table 8 are in line with the observations made in the previous section. Again, the marketing gains are higher under the S&P default probability rating system as compared to the Moody's expected default loss rating system. As illustrated by cases (ii) and (iii), the higher the systematic risk, β, and the smaller the residual risk, σ_ε, the higher is the marketing gain from securitisation.

Reducing the number of tranches from 6 to 2 (case iv) reduces the amount of marketing gain sharply for the default probability rating system. Under the expected default loss rating system the marketing gain can be increased when issuing only two tranches corresponding to the most senior and the most junior rating of the six tranche deal. This is due to the fact that if ratings are based on expected losses, deleting mezzanine tranches, enables the SPV to issue a higher share of non-senior debt (e.g. the market value of the B tranche in this case is bigger than the sum below AAA tranche values in the basic example), on which it is possible to make a substantial profit. For both rating systems it holds that the better the tranche rating, especially the better the rating of the lowest rated tranche, the smaller the amount of total marketing gain.

The number of bonds in the underlying portfolio (case v) has a negligible effect on the marketing gain which is probably due to the homogeneity of the bonds and the assumption of constant correlation. Varying the market parameters in cases (vi) and (vii) we observe that the greater the market risk premium $(r_m - r_f)$ and the higher the market volatility (σ_m), the greater is the marketing gain. However, the volatility effect is rather small.

Securitising a better quality portfolio leads to smaller gains because in this case the percentage volume of low rated tranches, yielding the highest gains, decreases as compared to the base case. As before, the risk characteristics of the reference firm have a revers effect, the smaller β^* and the higher σ_ε^*, the higher is the marketing gain. However, the effect are less pronounced than the effects from varying the characteristics of the corporate issuers.

The simulation results when assuming a fixed recovery rate of 40% on the bonds in the underlying portfolio do not deviate much from those derived within the Merton model with an endogeneous recovery rate. However, case (xi) shows that this result is quite sensitive to the assumed recovery rate. A higher recovery rate reduces the marketing gains since this reduces the risk of the underlying portfolio.

8 Conclusion

In this paper we have analyzed the gains from issuing tranched debt in a market in which structured bonds can be sold to investors at prices and yields that reflect only their credit rating. This rating can be based on default probabilities as in the case of Standard and Poor's or on expected default losses as in the case of Moody's. For both rating systems, we find general conditions under which tranched debt is overpriced. These conditions relate to the risk characteristics of the collateral relative to those of the reference firm from which rating-based bond yields are derived.

The CAPM asset pricing theory and the Merton (1974) structural debt model are used to value both corporate bonds and securitised tranches. We show that the marketing gains under both rating systems are highest when the systematic risk β of the collateral is high and the residual risk σ_ε is low relative to that of the reference firm. In all cases we find significant additional gains to multi-tranching, which is consistent with the fact that there were 5.58 tranches in the average securitisation in the US in 2003.[33] In every case, we find that the marketing gains from multiple tranches are significantly higher when the securities are valued using S&P ratings than when they are valued using Moody's ratings.

Many structured products are heavily marketed based on their credit ratings. Our analysis pinpoints the source and the level of mispricing when an investment banker sells CDO tranches at the same yield as an equally rated bond. Our analysis highlights the limitations of current credit rating systems which reflect characteristics of the *total* risk of fixed income securities, neglecting the more important price relevant risk characteristics, like systematic risk. If ratings are to be used for valuation then it is important that they reflect the systematic risk of the securities.

[33] See Cuchra and Jenkinson (2005).

A Proofs

A.1 Proof of Lemma 1

(a) If $P \geq^{FSD} P^*$, the first order stochastic dominance ranking of the physical distributions implies that under a default probability rating system $B_k \geq B_k^*$. Then note that (8) can be written as:

$$\begin{aligned} \mathbf{\Omega} &= \frac{B_k}{B_k^*} E_{Q^*}\left\{min[B_k^*, V^*]\right\} - E_Q\left\{min[B_k, V]\right\} & (25) \\ &= E_{Q^*}\left\{min[B_k, \frac{B_k}{B_k^*}V^*]\right\} - E_Q\left\{min[B_k, V]\right\} \\ &\geq E_{Q^*}\left\{min[B_k, V^*]\right\} - E_Q\left\{min[B_k, V]\right\} & (26) \end{aligned}$$

$\mathbf{\Omega}$ is positive if $Q^* \geq^{SSD} Q$.

For the converse argument note that $P^* \geq^{FSD} P$ implies $B_k < B_k^*$.

(b) Note that if $P_2 \geq^{FSD} P_1$ the face value of the k-rated bond issued by the second issuer, B_k^2, is greater than the face value of bond issued by the first issuer, B_k^1. This implies that Ω^2 is greater than Ω^1 since expression (25) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.2 Proof of Lemma 2

(a) If $P \geq^{SSD} P^*$, the second order stochastic dominance ranking of the physical distributions implies that under an expected default loss rating system $B_k \geq B_k^*$. The rest of the proof follows from the proof of Lemma 1.

(b) If $P_2 \geq^{SSD} P_1$ the face value of the k-rated bond issued by the second issuer, B_k^2, is greater than the face value of bond issued by the first issuer, B_k^1. This implies that Ω^2 is greater than Ω^1 since expression (25) is increasing in B_k for $\Omega \geq 0$, i.e. when $Q^* \geq^{SSD} Q$.

A.3 Proof of Lemma 4

$$\Delta\Omega = {\phi^*}_{k_1} B_{k_1} + {\phi^*}_{k_2} B_{k_2} - {\phi^*}_k B_k \tag{27}$$

Now

$$\phi^*_{k_1} \equiv \frac{E_{Q^*} min[B^*_{k_1}, V]}{B^*_{k_1}}, \ \phi^*_{k_2} \equiv \frac{E_{Q^*} min[B^*_{k_2}, V]}{B^*_{k_2}}, \ \phi^*_{k} \equiv \frac{E_{Q^*} min[B^*_{k}, V]}{B^*_{k}} \quad (28)$$

Therefore substituting from equations (28) in (27) and noting that $B_k = B_{1,k_1} + B_{2,k_2}$, we have:

$$\begin{aligned} \Delta\Omega &= \frac{B_{1,k_1}}{B^*_{k_1}} E_{Q^*} min[B^*_{k_1}, V] + \frac{B_{2,k_2}}{B^*_{k_2}} E_{Q^*} min[B^*_{k_2}, V] \\ &- \frac{B_{1,k_1} + B_{2,k_2}}{B^*_{k}} E_{Q^*} min[B^*_{k}, V] \end{aligned} \quad (29)$$

Now, under an expected default loss rating system, the SPV bonds have the same expected payoff per unit of face value as do the correspondingly rated corporate bonds, so that:

- for the untranched issue:

$$\frac{E_P min[B_k, V]}{B_k} = \frac{E_{P^*} min[B^*_k, V]}{B^*_k} \quad (30)$$

- for the senior tranche:

$$\frac{E_P min[B_{1,k_1}, V]}{B_{1,k_1}} = \frac{E_{P^*} min[B^*_{k_1}, V]}{B^*_{k_1}} \quad (31)$$

- for the junior tranche:

$$\frac{E_P \{min[B_k, V] - min[B_{1,k_1}, V]\}}{B_{2,k_2}} = \frac{E_{P^*} min[B^*_{k_2}, V]}{B^*_{k_2}} \quad (32)$$

Then substituting for B^*_k, $B^*_{k_1}$, and $B^*_{k_2}$ from equations (30)-(32) in (30):

$$\begin{aligned} \Delta\Omega &= \left\{ \frac{E_{Q^*} min[B^*_{k_1}, V]}{E_{P^*} min[B^*_{k_1}, V]} - \frac{E_{Q^*} min[B^*_{k_2}, V]}{E_{P^*} min[B^*_{k_2}, V]} \right\} E_P min[B_{1,k_1}, V] \\ &+ \left\{ \frac{E_{Q^*} min[B^*_{k_2}, V]}{E_{P^*} min[B^*_{k_2}, V]} - \frac{E_{Q^*} min[B^*_{k}, V]}{E_{P^*} min[B^*_{k}, V]} \right\} E_P min[B_k, V] \end{aligned} \quad (33)$$

Define the bond payoffs, $\pi^*_1(v) = min[B^*_{k_1}, v]$, $\pi^*_2(v) = min[B^*_{k_2}, v]$, $\pi^*(v) = min[B^*_k, v]$, $\pi_1(v) = min[B_{1,k_1}, v]$, $\pi_2(v) = min[B_{2,k_2}, v]$ and recall that $E_{Q^*}[v] = E_{P^*}[m^*(v)v]$. Then the incremental profit from the second tranche is

$$\begin{aligned} \Delta\Omega &= \left\{ \frac{E_{P^*}[m^* \pi^*_1]}{E_{P^*}[\pi^*_1]} - \frac{E_{P^*}[m^* \pi^*_2]}{E_{P^*}[\pi^*_2]} \right\} E_P[\pi_1] \\ &\quad + \left\{ \frac{E_{P^*}[m^* \pi^*_2]}{E_{P^*}[\pi^*_2]} - \frac{E_{P^*}[m^* \pi^*]}{E_{P^*}[\pi^*]} \right\} E_P[\pi_1 + \pi_2] \\ &= (E_P[\pi_1] + E_P[\pi_2]) E_{P^*}[m^*(v) w(v)] \end{aligned} \quad (34)$$

where

$$w_x(v) = x\left(\frac{\pi_1^*(v)}{E_{P^*}[\pi_1^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]}\right) + (1-x)\left(\frac{\pi_2^*(v)}{E_{P^*}[\pi_2^*(v)]} - \frac{\pi^*(v)}{E_{P^*}[\pi^*(v)]}\right) \quad (35)$$

and $x = E_P[\pi_1(v)]/(E_P[\pi_1(v)]+E_P[\pi_2(v)])$. A second tranche will be profitable if there exists an x such that $E_{P^*}[m^*(v)w_x(v)] > 0$. $w_x(v)$ is a piecewise linear function with slopes given by:

$$\frac{dw_x(v)}{dv} = \begin{cases} x\left[\frac{1}{E_{P^*}[\pi_1^*]} - \frac{1}{E_{P^*}[\pi_2^*]}\right] + \left[\frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]}\right] & for\, v < B_{k_1}^* & (i) \\ (1-x)\frac{1}{E_{P^*}[\pi_2^*]} - \frac{1}{E_{P^*}[\pi^*]} & for\, B_{k_1}^* < v < B_k^* & (ii) \\ (1-x)\frac{1}{E_{P^*}[\pi_2^*]} & for\, B_k^* < v < B_{k_2}^* & (iii) \\ 0 & for\, v > B_{k_2}^* & (iv) \end{cases}$$

Note that the face value and therefore the expected payoff of a corporate bond is a decreasing function of its rating so that:

$$\frac{1}{E_{P^*}[\pi_1^*]} > \frac{1}{E_{P^*}[\pi^*]} > \frac{1}{E_{P^*}[\pi_2^*]}$$

Then for $0 \leq x \leq 1$ the slope dw_x/dv is negative in region (ii), positive in region (iii) and zero in region (iv). Note that $E_{P^*}[w_x(v)] = 0$. Consider $x = \hat{x}$ such that $w_{\hat{x}}(v) = 0$ in region (iv). Equation (35) implies that

$$\hat{x} = \frac{B_k^*/E_{P^*}[\pi^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}{B_{k_1}^*/E_{P^*}[\pi_1^*(v)] - B_{k_2}^*/E_{P^*}[\pi_2^*(v)]}$$

Since $E_{P^*}[w_x(v)] = 0$, the slope conditions in regions (ii) and (iii) imply that $w_{\hat{x}}(v) > 0$ in region (i), which is sufficient for $\mathbf{\Delta\Omega} \propto E_{P^*}[m^*(v)w_x(v)] > 0$ if $m^*(v)$ is a decreasing function.

B Simulating SPV Cash Flows

In the following we sketch our simulation procedure.

1. *Determination of Debt Face Value*

   Given the rating k and maturity τ of a bond issued by firm j we can determine the nominal value, $\hat{B}_k$, of each bond in the SPV portfolio. Under the default probability rating system $\hat{B}_k$ is obtained from equation (11) using the historical default probability given by S&P.

Under the expected default loss rating system we have to solve equations (12) and (13) iteratively for $\hat{B}_k$ until the expected loss rate, Λ_k, equal to that given by the Moody's rating.[34]

2. *Simulation of SPV Value*

For each firm associated with the bonds in the SPV portfolio we can simulate its asset value at τ under the physical measure by:

$$\begin{aligned} V_j(\tau) &= V_j(0)\exp[(\mu - 0.5\sigma^2)\tau + \beta\sigma_m\sqrt{\tau}z_0 + \sigma_\varepsilon\sqrt{\tau}z_j] \\ & z_0, z_j \; iid\,\mathcal{N}(0,1) \quad j = 1,\ldots,J \end{aligned} \tag{36}$$

Analogously the risk-neutral value, $V_j^Q(\tau)$, is given by the same formula with μ replaced by r_f. For each simulation run n, $V_j(\tau)$ is produced for all J firms, and the cashflow from bond j can then be determined as

$$CF_{j,n}(\tau) = min[V_{j,n}(\tau), \hat{B}_k] \tag{37}$$

The bond defaults if $V_{j,n}(\tau) < \hat{B}_k$.[35]

The total portfolio cashflow under the physical measure is then given by

$$CF_{SPV,n}(\tau) = \sum_{j=1}^{J} CF_{j,n}(\tau) \tag{38}$$

and, analogously, under the risk-neutral measure

$$CF^Q_{SPV,n}(\tau) = \sum_{j=1}^{J} min[V^Q_{j,n}(\tau), \hat{B}_k] \tag{39}$$

Performing N simulation runs, we get the distribution of the portfolio value in τ under both measures. The market value of the portfolio at $t = 0$ is then derived as:

$$W_{SPV} = e^{-r_f\tau}\frac{1}{N}\sum_{n=1}^{N} CF^Q_{SPV,n}(\tau) \tag{40}$$

[34] In case of using a fixed recovery rate of R, meaning that the bond pays off $R \cdot \hat{B}_k$ in any default state, equation (13) reduces to $\hat{\mathcal{L}} = \hat{B}_k(1-R)\mathcal{N}(-d_2^{\hat{P}})$.

[35] In case of using a fixed recovery rate, equation (37) is replaced by $CF_{j,n}(\tau) = \hat{B}_k$ for $V_{j,n}(\tau) \geq \hat{B}_k$ and $CF_{j,n}(\tau) = R \cdot \hat{B}_k$ for $V_{j,n}(\tau) < \hat{B}_k$

3. *Tranche Valuation*

   We assume that the SPV issues I tranches with ratings k_i $(i = 1, \ldots, I)$ against the portfolio of bonds. Under the default probability rating system, the aggregate face value B_{k_i} for the SPV portfolio is determined by taking the Π_{k_i}- quantile of the physical distribution of the SPV value obtained from step 2. Again, B_{k_i} has to be solved iteratively under the expected default loss rating system.

   Given B_{k_i}, the total market value of the aggregate bond written on the SPV is then derived under the risk-neutral measure by

$$W_{k_i} = e^{-r_f T} \frac{1}{N} \sum_{n=1}^{N} min[CF^Q_{SPV,n}, B_{k_i}] \tag{41}$$

   The face and market values of each tranche are then calculated as the first differences of the aggregate values:

$$B_{i,k_i} = B_{k_i} - B_{k_{i-1}}, \tag{42}$$

$$W_{i,k_i} = W_{k_i} - W_{k_{i-1}}, \tag{43}$$

   with the first tranche, $B_{1,k_1} = B_{k_1}$ and $W_{1,k_1} = W_{k_1}$. The market value of the equity piece can then be derived as

$$W_{equity} = W_{SPV} - \sum_{i=1}^{I} W_{i,k_i} \tag{44}$$

4. *Sales Price and Profit*

   First the yield on the reference bonds with ratings k_i is determined. Given the risk characteristics (β^*, σ^*) of the reference firm on which ratings are based, we can again determine the face value, $B^*_{k_i}$, of the reference bond and the corresponding market value, $W^*_{k_i}$ according to Merton's formula as given by equation (16).[36] Then the yield is defined as

$$y^*_{k_i} = \frac{1}{T} \ln \frac{B^*_{k_i}}{W^*_{k_i}} \tag{45}$$

[36] Using the assumption of a fixed recovery rate R for the reference bond the value of this bond is given by $W^*_{k_i} = B^*_k e^{-r_f \tau} \mathcal{N}(d_2^{Q^*}) + R \cdot B^*_k e^{-r_f \tau} \mathcal{N}(-d_2^{Q^*})$

According to our pricing assumption, the sales price of tranche i is given by

$$S_{i,k_i} = e^{-y^*_{k_i}T} B_{i,k_i} \tag{46}$$

such that the profit on tranche i is derived as

$$\Omega_i = S_{i,k_i} - W_{i,k_i} \tag{47}$$

The total profit is given by $\mathbf{\Omega} = \sum \Omega_i$ which equals a percentage profit of $\frac{\mathbf{\Omega}}{W_{SPV}}$ on the portfolio'a market value.

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Table 1:
Cumulative Default Frequencies for Corporate Issues (Standard & Poor's 2005).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.02	0.03	0.06	0.10	0.14
AA	0.01	0.04	0.09	0.14	0.22	0.31	0.42
A	0.02	0.08	0.17	0.30	0.46	0.66	0.89
BBB	0.29	0.68	1.16	1.71	2.32	2.98	3.67
BB	2.30	4.51	6.60	8.57	10.42	12.18	13.83
B	5.30	10.83	15.94	20.48	24.46	27.95	31.00

The table reports historical cumulative default frequencies (in percent) for the period 1981 to 2003 for 9,740 companies of which 1,386 defaulted.

Table 2:
Cumulative Default Frequencies for CDO tranches (Standard & Poor's 2005).

	1	2	3	4	5	6	7
AAA	0.00	0.01	0.03	0.07	0.12	0.19	0.29
AA	0.01	0.06	0.14	0.23	0.36	0.51	0.70
A	0.03	0.12	0.26	0.46	0.71	1.01	1.37
BBB	0.35	0.83	1.41	2.07	2.81	3.61	4.44
BB	2.53	4.95	7.23	9.38	11.40	13.31	15.11
B	5.82	11.75	17.15	21.92	26.09	29.73	32.90

The table reports cumulative default frequencies (in percent) based on "quantitative and qualitative considerations" (Standard & Poor's 2005, p. 10).

Table 3: Cumulative 'Idealized Loss Rates' according to Moody's (2005).

	1	2	3	4	5	6	7
Aaa	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Aa	0.00	0.00	0.01	0.03	0.04	0.05	0.06
A	0.01	0.04	0.12	0.19	0.26	0.32	0.39
Baa	0.09	0.26	0.46	0.66	0.87	1.08	1.33
Ba	0.86	1.91	2.85	3.74	4.63	5.37	5.89
B	3.94	6.42	8.55	9.97	11.39	12.46	13.21

Table 4: Corporate Bond Valuation under the *Default Probability* rating system

Panel A: Valuation of Reference Bonds by Rating Class

i	S&P Rating (k_i)	Probability of Default Π_{k_i}	(Expected Loss) (Λ_{k_i})	Face Value $B^*_{k_i}$	Equilibrium Value $W^*_{k_i}$	Rating-based Bond Yield $y^*_{k_i}$	$\phi^*_k \equiv W^*_{k_i}/B^*_{k_i}$
1	AAA	0.061%	(0.009%)	18.02	15.12	3.51%	0.839
2	AA	0.219%	(0.033%)	22.81	19.12	3.53%	0.838
3	A	0.459%	(0.075%)	26.49	22.17	3.56%	0.837
4	BBB	2.323%	(0.440%)	38.59	31.96	3.77%	0.828
5	BB	10.424%	(2.416%)	60.47	47.90	4.66%	0.792
6	B	24.460%	(6.754%)	85.54	62.41	6.31%	0.730

Panel B: Valuation of Subordinated Bonds and Sales Prices by Rating Class

i	S&P Rating (k_i)	Probability of Default Π_{k_i}	(Expected Loss) (Λ_{k_i})	ϕ^*_k	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	AAA	0.061%	(0.009%)	0.839	18.02	18.02	15.12	15.12	3.51%	15.12	0.00
2	AA	0.219%	(0.128%)	0.838	22.81	4.79	19.12	4.00	3.60%	4.01	0.01
3	A	0.459%	(0.329%)	0.837	26.49	3.68	22.17	3.05	3.74%	3.08	0.03
4	BBB	2.323%	(1.239%)	0.828	38.59	12.10	31.96	9.79	4.24%	10.02	0.23
5	BB	10.424%	(5.902%)	0.792	60.47	21.88	47.90	15.94	6.34%	17.33	1.39
6	B	24.460%	(17.214%)	0.730	85.54	25.97	62.41	14.51	10.93%	18.29	3.78
-	Equity						100.00	37.59		37.59	0.00
									Total:	105.45	5.45

Parameter assumptions: $V^*(0) = V(0) = 100$, $\tau = 5$, $r_f = 3.5\%$, $r_m - r_f = 7\%$, $\sigma_m = 0.14$, $(\beta^*; \sigma^*_\varepsilon) = (\beta; \sigma_\varepsilon) = (0.8; 0.25)$

Table 5: Corporate Bond Valuation under the *Expected Default Loss* rating system

Panel A: Valuation of Reference Bonds by Rating Class

i	Moody's Rating (k_i)	Expected Loss Λ_{k_i}	(Probability of Default) (Π_{k_i})	Face Value $B^*_{k_i}$	Equilibrium Value $W^*_{k_i}$	Rating-based Bond Yield $y^*_{k_i}$	$\phi^*_k \equiv W^*_{k_i}/B^*_{k_i}$
1	Aaa	0.002%	(0.012%)	13.72	11.52	3.50%	0.839
2	Aa	0.037%	(0.241%)	23.24	19.48	3.53%	0.838
3	A	0.257%	(1.428%)	34.17	28.44	3.67%	0.832
4	Baa	0.869%	(4.273%)	45.56	37.33	3.98%	0.820
5	Ba	4.626%	(17.989%)	74.56	56.56	5.52%	0.759
6	B	11.390%	(36.730%)	106.15	71.42	7.93%	0.673

Panel B: Valuation of Subordinated Bonds and Sales Prices by Rating Class

i	Moody's Rating (k_i)	Exp. Loss Λ_{k_i}	(Probability of Default) (Π_{k_i})	ϕ^*_k	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	Aaa	0.002%	(0.012%)	0.839	13.72	13.72	11.52	11.52	3.50%	11.52	0.00
2	Aa	0.037%	(0.078%)	0.838	18.82	5.09	15.79	4.27	3.53%	4.27	0.00
3	A	0.257%	(0.533%)	0.832	27.34	8.52	22.87	7.08	3.69%	7.09	0.01
4	Baa	0.869%	(1.274%)	0.820	33.25	5.91	27.70	4.83	4.05%	4.85	0.02
5	Ba	4.626%	(9.262%)	0.759	58.03	24.78	46.27	18.56	5.78%	18.80	0.24
6	B	11.390%	(13.618%)	0.673	66.70	8.67	51.89	5.63	8.64%	5.83	0.20
-	Equity						100.00	48.11		48.11	0.00
									Total:	100.47	0.47

Parameter assumptions: $V^*(0) = V(0) = 100$, $\tau = 5$, $r_f = 3.5\%$, $r_m - r_f = 7\%$, $\sigma_m = 0.14$, $(\beta^*; \sigma^*_\varepsilon) = (\beta; \sigma_\varepsilon) = (0.8; 0.25)$

Table 6: Marketing Gains from Tranching Corpoarte Debt

Panel A: Under a *Default Probability Rating System*

Corpoarte Issuer			Five Tranches			Six Tranches		
β	σ_ε	Lemma 1 (a)	Total Debt	Ω^M_{BB}	Ω^S_{BB}	Total Debt	Ω^M_B	Ω^S_B
0.5	0.15	x	67.1	1.58	-1.24	78.3	4.56	-3.47
	0.25		46.5	0.90	-0.74	60.5	3.31	-1.94
	0.35		30.3	0.54	-0.36	44.2	2.41	-0.84
0.8	0.15		67.4	2.96	0.18	79.2	7.84	-0.27
	0.25		**47.9**	**1.67**	**0.00**	**62.4**	**5.45**	**0.00**
	0.35		31.7	0.96	0.03	46.1	3.79	0.36
1.1	0.15		65.8	4.34	1.71	78.4	11.19	3.23
	0.25		48.1	2.53	0.88	63.1	7.82	2.31
	0.35	✓	32.3	1.45	0.51	47.3	5.33	1.81
Reference Firm								
β^*	σ^*_ε							
1.1	0.25		47.9	1.29	-0.86	62.4	4.43	-2.20
0.5	0.25		47.9	2.00	0.77	62.4	6.39	2.07
0.8	0.15		47.9	1.60	-0.13	62.4	5.44	0.22
0.8	0.35		47.9	1.66	-0.05	62.4	5.30	-0.48

Panel B: Under a *Expected Loss Rating System*

Corpoarte Issuer			Five Tranches			Six Tranches		
β	σ_ε	Lemma 2 (a)	Total Debt	Ω^M_{Ba}	Ω^S_{ba}	Total Debt	Ω^M_B	Ω^S_b
0.5	0.15		66.5	0.32	-0.08	69.8	0.46	-0.12
	0.25	x	45.0	-0.31	-0.57	50.3	-0.41	-0.86
	0.35	x	28.5	-0.43	-0.57	34.2	-0.68	-0.97
0.8	0.15	✓	66.5	1.44	1.04	70.4	1.97	1.35
	0.25		**46.3**	**0.26**	**0.00**	**51.9**	**0.47**	**0.00**
	0.35		29.7	-0.15	-0.30	35.8	-0.19	-0.50
1.1	0.15	✓	65.7	2.71	2.30	68.2	3.26	2.72
	0.25	✓	46.2	0.91	0.65	52.3	1.48	1.00
	0.35		30.2	0.19	0.04	36.6	0.41	0.09
Reference Firm								
β^*	σ^*_ε							
1.1	0.25	x	46.3	-0.22	-0.63	51.9	-0.24	-0.94
0.5	0.25	✓	46.3	0.74	0.60	51.9	1.18	0.93
0.8	0.15		46.3	-0.34	-0.82	51.9	-0.36	-1.18
0.8	0.35	✓	46.3	0.58	0.41	51.9	0.93	0.62

The table shows the marketing gains from tranching debt into a five (six) tranches with ratings AAA, AA, A, BBB, BB (and B) when $r_f = 3.5\%$, $r_m - r_f = 7\%$ and $\sigma_m = 0.14$. First, the characteristics of the reference firm $(\beta^*, \sigma^*_\varepsilon) = (0.8, 0.25)$ are fixed and the systematic and idiosyncratic risk parameters $(\beta, \sigma_\varepsilon)$ of the arbitrary corporate issuer are varied. The last four line in each Panel show the reverse case holding $(\beta, \sigma_\varepsilon) == (0.8, 0.25)$ fixed. Lemmas 1(a) and 2(a) provide sufficient conditions for a gain (✓) or a loss (x) from a issuing single debt. The total amount of debt is the sum of the equilibrium market values of the overall debt issue. Ω^M_{BB} (Ω^M_B) is the marketing gain from a five (six) tranche securitisation expressed as percent of the underlying collateral value. Ω^S_{BB} (Ω^S_B) is the marketing gain from a *single* debt issue with the same total amount of debt as the corresponding multi-tranche securitisation. Note that unlike under the default probability rating system the rating of the single debt issue under the expected default loss rating system is no longer Ba (B). The numbers presented in bold fonts correspond to the basic examples presented in Tables 4 and 5.

Table 7: Pricing the liabilities of a 5-year maturity SPV holding corporate bond collateral

Panel A: Default Probability Rating System

i	S&P Rating (k_i)	Probability of Default Π_{k_i}	(Expected Loss) (Λ_{k_i})	ϕ_k^*	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	AAA	0.061 %	(0.00%)	0.839	93.46	93.46	78.37	78.37	3.52%	78.42	0.06
2	AA	0.219 %	(0.16%)	0.838	98.95	5.49	82.84	4.48	4.10%	4.61	0.13
3	A	0.459 %	(0.42%)	0.837	101.61	2.66	84.96	2.12	4.54%	2.23	0.11
4	BBB	2.323 %	(1.47%)	0.828	110.74	9.12	91.72	6.76	6.00%	7.56	0.80
5	BB	10.424 %	(7.15%)	0.792	118.89	8.15	96.53	4.81	10.57%	6.46	1.65
6	B	24.46 %	(21.23%)	0.730	124.44	5.55	98.69	2.16	18.88%	4.05	1.89
-	Equity							1.31		1.31	
	Total						100.00			104.63	4.63

Panel B: Expected Default Loss Rating System

i	Moody's Rating (k_i)	Exp. Loss Λ_{k_i}	(Probability of Default) (Π_{k_i})	ϕ_k^*	Face Value Cumulative B_{k_i}	Face Value Tranche B_{i,k_i}	Equilibrium Value Cumulative W_{k_i}	Equilibrium Value Tranche W_{i,k_i}	Equilibrium Yield to Maturity	Sales Price S_{i,k_i}	Gain
1	Aaa	0.002%	(0.01%)	0.839	80.82	80.82	67.81	67.81	3.51%	67.85	0.04
2	Aa	0.037%	(0.14%)	0.836	90.31	9.49	75.65	7.84	3.82%	7.94	0.10
3	A	0.257%	(0.48%)	0.834	96.17	5.86	80.35	4.70	4.40%	4.88	0.18
4	Baa	0.869%	(1.38%)	0.818	103.83	7.66	86.16	5.81	5.54%	6.27	0.46
5	Ba	4.626%	(10.37%)	0.759	117.92	14.09	94.84	8.69	9.67%	10.69	2.00
6	B	11.390%	(12.53%)	0.672	119.67	1.75	95.66	0.82	15.33%	1.18	0.36
-	Equity							4.34		4.34	
	Total						100.00	100.00		103.14	3.14

Parameter Assumptions: Collateral: 125 issues of B rated bonds; for issuers $(\beta;\sigma_\varepsilon) = (0.8; 0.25)$, $\tau = 5$. The implied correlation between issuer returns is 0.17. The asset risk of the firm underlying the ratings is $(\beta^*;\sigma_\varepsilon^*) = (0.8; 0.25)$ with $r_f = 3.5\%$, $r_m - r_f = 7\%$;$\sigma_m = 14\%$.

Table 8: Marketing Gains from Securitisation of Corporate Bonds

Example	Variation		Merton Model S&P Ratings	Merton Model Moody's Ratings	Fixed Recovery (40%) S&P Ratings	Fixed Recovery (40%) Moody's Ratings
(i)	Base Case		4.63%	3.14%	5.19%	3.09%
(ii)	$\beta(issuers)$	1.0	6.38%	4.14%	6.58%	4.05%
		0.8	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		0.7	3.79%	2.61%	4.19%	2.74%
(iii)	$\sigma_\varepsilon(issuers)$	0.30	4.07%	2.61%	4.01%	2.58%
		0.25	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		0.20	5.36%	3.73%	6.58%	4.05%
(iv)	Number of Tranches	2	0.97%	1.76%	0.48%	1.47%
		6	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		2	2.34%	5.37%	0.52%	5.42%
(v)	Number of Bonds	62	4.57%	3.12%	4.75%	3.03%
		125	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		140	4.62%	3.16%	5.04%	3.06%
(vi)	$r_m - r_f$	8%	5.62%	4.09%	6.45%	4.00%
		7%	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		6%	3.74%	2.34%	4.05%	2.33%
(vii)	σ_m	12%	4.53%	2.74%	4.40%	2.63%
		14%	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		16%	4.72%	3.60%	5.82%	3.77%
(viii)	Rating of Underlying	BB	2.40%	1.95%	3.47%	2.13%
		B	**4.63%**	**3.14%**	**5.19%**	**3.09%**
(ix)	β^*	1.0	4.42%	2.89%	4.73%	2.85%
		0.8	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		0.6	4.83%	3.39%	5.66%	3.37%
(x)	σ_ε^*	0.30	4.62%	3.28%	5.50%	3.26%
		0.25	**4.63%**	**3.14%**	**5.19%**	**3.09%**
		0.20	4.63%	2.94%	4.73%	2.84%
(xi)	Recovery Rate	20%	-	-	5.93%	3.96%
		40%	**-**	**-**	**5.19%**	**3.09%**
		60%	-	-	3.85%	2.15%

The table reports the marketing gains from securitising a portfolio corporate bonds when tranches are sold at rating-based yields according to S&P and Moody's ratings. The marketing gains are expressed as a *per cent* of the collateral value. The characteristics of the reference firm are set to $(\beta^*, \sigma_\varepsilon^*) = (0.8, 0.25)$; these parameters are varied in examples (ix) and (x). In addition, $r_f = 3.5\%$ and $r_m - r_f = 7.0\%$, $\sigma_m = 14.0\%$.
For the base case, the SPV holds a portfolio of 125 B-rated bonds whose issuers are characterized by the risk parameters $(\beta, \sigma_\varepsilon) = (0.8, 0.25)$. The SPV is assumed to issue 6 differently rated tranches corresponding to the ratings whose characteristics are described in Tables 1 and 3. In example (iv) the two tranches are first assumed to be rated AAA (Aaa) and BBB (Baa) and second AAA (Aaa) and B (Ba) by S&P (Moody's). For purpose of comparison the parameter and marketing gain of the base case are repeated in bold for each parameter perturbation.
The last two columns show the results when assuming a fixed recovery rate of 40% if a bond in the underlying portfolio defaults. This assumption is varied in case (xi).